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DIMENSIONAL ANALYSIS APPLIED TO THE WIND DISTRIBUTION IN THE PLANETARY BOUNDARY LAYER *

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ABSTRACT

In an attempt to understand the implications of Long's "generalized dimensional analysis," this method was applied to the problem of the wind distribution in the planetary boundary layer. The assumption was made that the equations of motion, together with an appropriate set of boundary conditions, define a unique relationship among the wind, the stress, and certain other variables and parameters. This relationship was found to be more precisely specified by the generalized analysis than by ordinary dimensional methods, although when the solution is required to reduce to the logarithmic wind profile near the ground both procedures give identical results and yield a universal relationship among the latitude, the surface roughness, the stress, the geostrophic wind, and the depth of the planetary layer, which is remarkably similar to one found by Rossby and Montgomery by a completely different argument. That such a result may be found by purely dimensional reasoning is taken as an indication of the power of the dimensional method.

The solution achieved was possible only through the artifice of treating all vectors as if they were vectors in two dimensions only. The dimensional method in its present form does not appear capable of treating the more complete vector problem, although there are indications that when the foundations of dimensional analysis, which lie in invariance theory, are better understood, such problems too will lend themselves to solution by dimensional reasoning.

1. INTRODUCTION

In a recent paper, Long [7] suggested that the most efficient use of dimensional analysis in fluid mechanics is in combination with the known mathematical form of the governing equations when these equations are not analytically solvable. This approach, which Long calls "generalized dimensional analysis," makes use of mathematical principles concerning the invariance of equations under transformations of the variables and completely ignores physical dimensions, considering only the mathematical relationships required by the governing equations. Solutions found by this method are at least as efficient (in the sense that they contain a minimum number of dimensionless variables) and frequently more efficient than those found by the more usual method which makes use of physical dimensions.

This paper describes an attempt to apply both Long's procedure and the ordinary dimensional method to the wind distribution in the planetary boundary layer, making use only of the equations of motion and the known logarithmic wind profile near the ground. Certain questions concerning the applicability of dimensional methods to vector problems, which arose in this study, will be discussed briefly. The dimensional method itself will not be reviewed here; it is discussed in a number of texts, the standard reference being that by Bridgman [2], while a more recent and more complete discussion is given by Langhaar [6]. A stimulating discussion of fundamental dimensions, including the concepts of vector lengths and the dual role of mass, is given by Huntley [4]. The notion of dimensional analysis as a special case of invariance

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theory is introduced by Langhaar and by Ipsen [5], and is discussed in more detail by Birkhoff [1] and Long [7, 8].

2. FORMULATION OF THE PROBLEM

The mathematical foundation of dimensional analysis lies in the Pi Theorem (Buckingham [3]; see also Langhaar [6], pp. 47–58) which states that if among the variables $x_1, x_2, \ldots, x_N$ there exists a functional relationship

$$\phi\{x_1, x_2, \ldots, x_N\}=0 \tag{1}$$

then there exists another functional relationship

$$\phi\{\pi_1, \pi_2, \ldots, \pi_n\}=0 \tag{2}$$

where $\pi_1, \pi_2, \ldots, \pi_n$ is a complete set of independent dimensionless combinations of the variables $x_1, x_2, \ldots, x_N$, and $(N-n)$ is the number of independent dimensions in terms of which $x_1, x_2, \ldots, x_N$ are expressed. (Throughout this paper, the symbol ϕ will refer to any unspecified function arising in a dimensional argument; ϕ does not necessarily represent the same function in the various equations in which it appears.) In ordinary dimensional analysis, the variables $x_1, x_2, \ldots, x_N$ are expressed in terms of the physical dimensions, mass, length, time, etc. The generalized approach, on the other hand, ignores physical dimensions and seeks the maximum number of independent dimensions that will produce dimensional homogeneity in the governing equations. Thus for the generalized procedure to be applicable the governing equations must be known.

A problem which seems to lend itself to this treatment is that of the wind distribution in the planetary boundary layer. The problem may be stated in the following way. Very near the ground, the mean wind under adiabatic conditions is given by

$$u=\frac{u_*}{k}\ln\frac{z}{z_0} \tag{3}$$

where u_* is equal to $(\tau/\rho)^{1/2}$, k is the von Kármán constant, roughly equal to 0.4, and z_0 is the "roughness parameter" representing the height at which the mean wind becomes zero. (The Reynolds stress τ, and consequently u_*, is assumed constant with height within the immediate surface layer.) No such simple relationship is known for greater heights within the planetary boundary layer. There the mean flow is governed by the equations derived by Reynolds [9] in 1895 for a turbulent fluid of uniform density,

$$\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv+\frac{1}{\rho}\frac{\partial \tau_x}{\partial z}=f(v-v_g)+\frac{1}{\rho}\frac{\partial \tau_x}{\partial z}$$

$$\frac{dv}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu+\frac{1}{\rho}\frac{\partial \tau_y}{\partial z}=-f(u-u_g)+\frac{1}{\rho}\frac{\partial \tau_y}{\partial z} \tag{4}$$

where f is the Coriolis parameter, τ_x and τ_y represent Reynolds stresses, u_g and v_g are components of the geostrophic wind, and the terms representing molecular viscosity have been neglected in comparison with the terms representing turbulence. A third equation describes the vertical acceleration, but it is generally assumed to reduce to the hydrostatic equation and will be omitted in the present study. If there are no accelerations, the left sides of the equations become zero, and the wind components are given by

$$v=v_g-\frac{1}{\rho f}\frac{\partial \tau_x}{\partial z}$$

$$u=u_g+\frac{1}{\rho f}\frac{\partial \tau_y}{\partial z} \tag{5}$$

These equations relate the wind to the vertical gradient of the Reynolds stress, and are therefore not directly comparable with equation (3) which relates the wind to the stress itself.

The object of this study was to derive, by dimensional methods, an expression for the wind throughout the planetary layer that reduces to the logarithmic law near the ground. In order to do this it was necessary to make a number of assumptions whose validity may be questioned; these assumptions will be introduced and assessed later in light of the results obtained. First, however, it is important to note a basic distinction between the layer immediately above the surface and the remainder of the planetary boundary layer above it.

Very near the surface, the wind, wind shear, and stress all lie in the same direction, and the governing equation (3) is a scalar equation. Above the surface layer, however, the effects of the earth's rotation enter, the wind direction changes with height, and the problem becomes essentially vectorial. It is therefore natural to write the governing equations in vector form; however dimensional analysis would then introduce dimensionless "ratios" of vectors, such as $\mathbf{V}/\mathbf{V}_g$, and such "ratios" are not defined within ordinary vector usage. No such problem arises if the equations are retained in scalar component form; however it is then difficult to derive a single equation reducing to the logarithmic law near the ground. A third alternative, which will be adopted here, is to represent the vectors as complex variables; this is permissible since to a high degree of approximation all the relevant vectors lie in the horizontal plane and are therefore two-dimensional. By this artifice the essential vector aspects of the problem are retained, and since ratios of complex variables are defined, the usual dimensional methods may be applied. For simplicity in notation, the boldface type usually reserved for vectors will be used to represent complex variables; thus $\mathbf{V}$ will mean $u+iv$, $\boldsymbol{\tau}$ will mean $\tau_x+i\tau_y$, etc.; equation (4) then becomes

$$\frac{d\mathbf{V}}{dt}=-if(\mathbf{V}-\mathbf{V}_g)+\frac{1}{\rho}\frac{\partial \boldsymbol{\tau}}{\partial z} \tag{6}$$

and if there are no accelerations, this reduces to

$$\mathbf{V}=\mathbf{V}_g-\frac{i}{\rho f}\frac{\partial\tau}{\partial z} \tag{7}$$

The logarithmic law may be written in the same notation as

$$\mathbf{V}=\frac{\mathbf{V}_*}{k}\ln\frac{z}{z_0} \tag{8}$$

where $\mathbf{V}_*$ is defined as $(\tau/\rho)^{1/2}$. The assumption is now made that there exists an equation, valid throughout the planetary layer, of the form

$$\mathbf{V}=\mathbf{V}_*\phi\left\{\frac{z}{z_0}, \alpha, \beta, \ldots\right\} \tag{9}$$

where ϕ represents an unspecified function and $\alpha, \beta, \ldots$ are whatever additional variables and parameters may be required. Further it is assumed that this equation is uniquely determined by the equation of motion (7) together with an appropriate set of boundary conditions, and that this equation reduces to the logarithmic law (8) as z approaches z_0. For the sake of simplicity it will be assumed that the geostrophic wind does not vary with height (i.e., that the atmosphere is barotropic); a similar but more complicated analysis may be performed for the more general baroclinic case. For our boundary conditions we note that at the top of the planetary layer the wind shear and the stress vanish and the wind becomes geostrophic, while at the bottom of the planetary layer the wind itself vanishes, and we write

$$\begin{aligned}
&\text{(a)} \quad \lim_{z\to z_0}\mathbf{V}=0\\
&\text{(b)} \quad \lim_{z\to z_h}\mathbf{V}=\mathbf{V}_g\\
&\text{(c)} \quad \lim_{z\to z_0}\tau=\tau_0\\
&\text{(d)} \quad \lim_{z\to z_h}\tau=0
\end{aligned} \tag{10}$$

where z_h represents the top, and z_0 the bottom, of the planetary layer. Other boundary conditions could conceivably be written, e.g.,

$$\lim_{z\to z_h}\frac{\partial\mathbf{V}}{\partial z}=0;\ \lim_{z\to z_h}\frac{\partial\tau}{\partial z}=0$$

but although these are true statements, they do not appear to be relevant to the problem and have therefore been excluded, the aim being to write the minimum number of equations needed to provide a solution.

Two different procedures may be used to find a solution to this problem. First the ordinary dimensional method will be considered, taking the set of relevant variables to be those variables appearing in the governing equations (7) and (10). Then Long's generalized method will be introduced, and the differences between the solutions found by the two methods will be assessed.

3. ANALYSIS BY THE ORDINARY DIMENSIONAL METHOD

From equations (7) and (10) the variables relevant to flow in the planetary layer are seen to be $\mathbf{V}$, $\mathbf{V}_g$, ρ, f, τ, τ_0, z, z_h, and z_0. Accordingly we assume that a functional relationship exists of the form

$$\phi\{\mathbf{V}, \mathbf{V}_g, \rho, f, \tau, \tau_0, z, z_h, z_0\}=0 \tag{11}$$

and we seek the appropriate set of independent dimensionless combinations. The physical dimensions of the variables are

$$\begin{aligned}
&[\mathbf{V}]=[\mathbf{V}_g]=LT^{-1}\\
&[\tau]=[\tau_0]=ML^{-1}T^{-2}\\
&[z]=[z_h]=[z_0]=L\\
&[\rho]=ML^{-3}\\
&[f]=T^{-1}
\end{aligned} \tag{12}$$

and these nine variables in three dimensions yield six independent dimensionless combinations, permitting us to write

$$\phi\left\{\frac{\mathbf{V}}{(\tau/\rho)^{1/2}}, \frac{\mathbf{V}}{\mathbf{V}_g}, \frac{\mathbf{V}}{fz_0}, \frac{z_h}{z_0}, \frac{z}{z_0}, \frac{\tau}{\tau_0}\right\}=0 \tag{13}$$

The dimensionless products may be rewritten in many ways, so long as care is taken to ensure that the set remains independent; thus instead of $\mathbf{V}/\mathbf{V}_g$ we may write $(\tau_0/\rho)^{1/2}/\mathbf{V}_g$, and instead of $\mathbf{V}/fz_0$ we may write $\mathbf{V}_g/fz_0$. Applying this procedure, introducing the symbol $\mathbf{V}_*$ for $(\tau/\rho)^{1/2}$, and solving for $\mathbf{V}$ we find

$$\mathbf{V}=\mathbf{V}_*\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}, \frac{\mathbf{V}_g}{fz_0}, \frac{z_h}{z_0}, \frac{z}{z_0}, \frac{\mathbf{V}_*}{\mathbf{V}_{*0}}\right\} \tag{14}$$

For reasons which will appear below, this solution has been written in such a way as to distinguish between parameters and variables; thus in any given situation $\mathbf{V}_{*0}/\mathbf{V}_g$, $\mathbf{V}_g/fz_0$ and z_h/z_0 are parameters, while z/z_0 and $\mathbf{V}_*/\mathbf{V}_{*0}$ are height-dependent variables.

We now invoke the condition that equation (14) must reduce to the logarithmic law near the ground. When z is small, $\mathbf{V}_*$ is equal to $\mathbf{V}_{*0}$ so that the ratio of these quantities is unity and (14) becomes

$$\mathbf{V}=\mathbf{V}_{*0}\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}, \frac{\mathbf{V}_g}{fz_0}, \frac{z_h}{z_0}, \frac{z}{z_0}\right\} \tag{15}$$

Setting (8) equal to (15) we have

$$\frac{1}{k}\ln\frac{z}{z_0}=\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}, \frac{\mathbf{V}_g}{fz_0}, \frac{z_h}{z_0}, \frac{z}{z_0}\right\} \tag{16}$$

Since z/z_0 is the only variable appearing in this equation (all the other dimensionless products being parameters), the right hand side must vary as $\ln(z/z_0)$ so that

$$\frac{1}{k}\ln\frac{z}{z_0}=\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g},\frac{\mathbf{V}_z}{fz_0},\frac{z_h}{z_0}\right\}\cdot\ln\frac{z}{z_0} \tag{17}$$

from which it follows that

$$\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g},\frac{\mathbf{V}_g}{fz_0},\frac{z_h}{z_0}\right\}=0 \tag{18}$$

or

$$\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}=\phi\left\{\frac{\mathbf{V}_g}{fz_0},\frac{z_h}{z_0}\right\} \tag{19}$$

At this point we may note that Rossby and Montgomery [10], in an extensive study of winds in the friction layer, concluded that under adiabatic conditions any one of $\mathbf{V}_{*0}/\mathbf{V}_g$, $\mathbf{V}_g/fz_0$, z_h/z_0, serves to determine the other two; thus

$$\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}=\phi_1\left\{\frac{\mathbf{V}_g}{fz_0}\right\}=\phi_2\left\{\frac{z_h}{z_0}\right\} \tag{20}$$

Rossby and Montgomery give the actual functions ϕ_1, and ϕ_2; see their equations (31b), (32a), (32b), (35). If their results are correct, then certainly the weaker requirement of equation (19) is satisfied.

Equation (18) states that any two of $\mathbf{V}_{*0}/\mathbf{V}_g$, $\mathbf{V}_g/fz_0$, z_h/z_0 serve to determine the third; consequently any one of these dimensionless products may be eliminated from the general solution (14). Thus if we choose to think of the surface stress as being completely determined by the geostrophic wind, the surface roughness, the depth of the friction layer, and the latitude, we may eliminate it by rewriting (14) as

$$\mathbf{V}=\mathbf{V}_*\phi\left\{\frac{\mathbf{V}_g}{fz_0},\frac{z_h}{z_0},\frac{z}{z_0},\frac{\mathbf{V}_*}{\mathbf{V}_g}\right\} \tag{21}$$

Alternately, we may reason that of all the parameters listed, z_h is the least susceptible to direct measurement and should therefore be excluded, and we may rewrite (14) as

$$\mathbf{V}=\mathbf{V}_*\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g},\frac{\mathbf{V}_g}{fz_0},\frac{z}{z_0},\frac{\mathbf{V}_*}{\mathbf{V}_{*0}}\right\} \tag{22}$$

As a third possibility, the effect of latitude may be excluded by writing

$$\mathbf{V}=\mathbf{V}_*\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g},\frac{z_h}{z_0},\frac{z}{z_0},\frac{\mathbf{V}_*}{\mathbf{V}_{*0}}\right\} \tag{23}$$

Thus equation (14), which represents the solution for an atmosphere governed solely by equations (7) and (10), reduces to (21), (22), or (23) when we introduce the condition that the logarithmic profile is the limiting case near the ground. We now proceed to compare these results with those obtained by generalized dimensional analysis.

4. ANALYSIS BY THE GENERALIZED DIMENSIONAL METHOD

The essential difference between ordinary and generalized dimensional analysis lies in the method of ascribing dimensions to the relevant variables. The ordinary method makes use of physical dimensions such as mass, length, time, etc., whereas the generalized method makes use of mathematical relationships among the variables specified by the governing equations. For example, if the geostrophic wind equation

$$u_g=-\frac{1}{\rho f}\frac{\partial p}{\partial y} \tag{24}$$

were one of the equations governing some problem, the ordinary method would ascribe dimensions as

$$\begin{aligned}[][u_g]&=LT^{-1}\\ [\rho]&=ML^{-3}\\ [f]&=T^{-1}\\ [p]&=ML^{-1}T^{-2}\\ [y]&=L\end{aligned} \tag{25}$$

There would thus be five variables in three dimensions, and two independent dimensionless products,

$$\frac{u_g^2}{fy} \text{ and } \frac{p}{\rho u_g^2}$$

would be formed. The generalized method would ascribe arbitrary dimensions $A, B, C, \ldots$ to the variables so that

$$\begin{aligned}[][u_g]&=A\\ [\rho]&=B\\ [f]&=C\\ [p]&=D\\ [y]&=E\end{aligned} \tag{26}$$

Then, for the equation to be dimensionally homogeneous, it is necessary that

$$[u_g]=\frac{[p]}{[\rho]\cdot[f]\cdot[y]} \tag{27}$$

so that

$$A=\frac{D}{BCE} \tag{28}$$

and D is equal to $ABCE$. The five variables are thus expressible in four independent dimensions, and only one independent dimensionless product arises,

$$\frac{p}{\rho f y u_g}$$

Since it is desirable to find the smallest possible number of independent dimensionless products, it would clearly be advantageous to use the generalized procedure in this case.

Long has pointed out that the generalized method is equivalent to the search for invariance of an equation under a set of transformations of the variables of the form $x=kx'$, k being a constant. Thus if the variables in (24) are transformed by

$$\begin{aligned} u_g&=\alpha u_g' \\ \rho&=\beta\rho' \\ f&=\gamma f' \\ p&=\delta p' \\ y&=\epsilon y' \end{aligned} \tag{29}$$

where α, β, γ, δ, and ϵ are constants, then the equation may be written in terms of the primed variables as

$$u_g'=\frac{\delta}{\alpha\beta\gamma\epsilon}\cdot\frac{1}{\rho' f'}\frac{\partial p'}{\partial y'} \tag{30}$$

This is identical in form to the original equation only if

$$\delta=\alpha\beta\gamma\epsilon \tag{31}$$

Thus the transformation constants α, β, γ, δ, and ϵ play the same role as the generalized dimensions A, B, C, D, and E.

Occasionally it appears that the generalized procedure yields fewer dimensions than the ordinary procedure. For instance, the physical dimensions of the variables appearing in the equation of state

$$p=\rho RT \tag{32}$$

are mass, length, time, and temperature, since

$$\begin{aligned} [p]&=ML^{-1}T^{-2} \\ [\rho]&=ML^{-3} \\ [R]&=L^2T^{-2}\theta^{-1} \\ [T]&=\theta \end{aligned} \tag{33}$$

Yet the generalized method shows that there are only three independent dimensions, for

$$[p]=[\rho]\cdot[R]\cdot[T] \tag{34}$$

This apparent contradiction is resolved when we see that the four physical dimensions in this example are not independent, and that the variables can be expressed in three independent dimensions as

$$\begin{aligned} [p]&=(ML^{-3})(LT^{-1})^2=DV^2 \\ [\rho]&=(ML^{-3})=D \\ [R]&=(LT^{-1})^2\theta^{-1}=V^2\theta^{-1} \\ [T]&=\theta \end{aligned} \tag{35}$$

where D is the dimension of density, V is the dimension of velocity, and θ is the dimension of temperature.

Generalized dimensional analysis can only be used when the governing equations are known, and is therefore not applicable to many problems that do lend themselves to ordinary dimensional methods. It is, however, applicable to the present problem. If arbitrary dimensions A, B, C, . . . are ascribed to the variables appearing in equations (5) and (8) such that

$$\begin{aligned} [\mathbf{V}]&=A & [\tau]&=E \\ [\mathbf{V}_g]&=B & [\tau_0]&=F \\ [\rho]&=C & [z]&=G \\ [f]&=D & [z_h]&=H \\ [z_0]&=I \end{aligned} \tag{36}$$

and the various equations are examined for dimensional homogeneity, it becomes apparent that

$$A=B=\frac{E}{CDG};\ E=F;\ G=H=I \tag{37}$$

and thus the nine variables may be expressed in terms of four independent dimensions (for example, C, D, E, and G) rather than three as found earlier by considering physical dimensions. Consequently we may form only five independent dimensionless combinations, and in place of equation (13) we have

$$\phi\left\{\frac{\tau_0}{\rho f z_0\mathbf{V}_g},\frac{\mathbf{V}}{\mathbf{V}_g},\frac{z_h}{z_0},\frac{z}{z_0},\frac{\tau}{\tau_0}\right\}=0 \tag{38}$$

We now introduce the symbol $\mathbf{V}_*$ for convenience, but since we are ignoring physical dimensions we know only that $\mathbf{V}_*^2$ has the same dimensions as $fz_0\mathbf{V}_g$; we do not know that $\mathbf{V}_*$ has the same dimensions as $\mathbf{V}$ or that fz_0 has the same dimensions as $\mathbf{V}_g$. Consequently we cannot express the solution for $\mathbf{V}$ as the product of $\mathbf{V}_*$ and some unspecified function as we did in (14), but must express it in some other way; one such expression is

$$\mathbf{V}=\mathbf{V}_g\phi\left\{\frac{\mathbf{V}_{*0}^2}{fz_0\mathbf{V}_g},\frac{z_h}{z_0},\frac{z}{z_0},\frac{\mathbf{V}_*}{\mathbf{V}_{*0}}\right\} \tag{39}$$

For purposes of comparison, equation (14) may be rewritten as

$$\mathbf{V}=\mathbf{V}_g\phi\left\{\frac{\mathbf{V}_{*0}}{\mathbf{V}_g},\frac{\mathbf{V}_g}{fz_0},\frac{z_h}{z_0},\frac{z}{z_0},\frac{\mathbf{V}_*}{\mathbf{V}_{*0}}\right\} \tag{40}$$

Comparing these solutions we see that not only does (39) contain one less dimensionless product, but it specifically indicates that the two quantities

$$\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}\ \text{and}\ \frac{\mathbf{V}_g}{fz_0}$$

which appear as independent variables in (40), do not affect the problem separately but only in the combination

$$\left(\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}\right)^2\cdot\left(\frac{\mathbf{V}_g}{fz_0}\right)=\frac{\mathbf{V}_{*0}^2}{fz_0\mathbf{V}_g} \tag{41}$$

which appears in (39). Thus the solution found by the generalized procedure, considering only equations (7) and (10) as the governing equations, is more efficient (since it contains fewer dimensionless products) than that found by the ordinary method.

We now introduce the condition that (39) must reduce to the logarithmic law near the ground. Proceeding as before, we set (8) equal to (39) with $\mathbf{V}_*$ equal to $\mathbf{V}_{*0}$, giving

$$\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}\cdot\frac{1}{k}\ln\frac{z}{z_0}=\phi\left\{\frac{\mathbf{V}_{*0}^2}{fz_0\mathbf{V}_g},\frac{z_h}{z_0},\frac{z}{z_0}\right\} \tag{42}$$

As before, z/z_0 is the only variable and may be eliminated from both sides, leaving

$$\frac{\mathbf{V}_{*0}}{\mathbf{V}_g}=\phi\left\{\frac{\mathbf{V}_{*0}^2}{fz_0\mathbf{V}_g},\frac{z_h}{z_0}\right\} \tag{43}$$

which is equivalent to (18) and (19). Thus the condition under which the generalized solution reduces to the logarithmic law near the ground is the same as for the ordinary solution. Furthermore, if equation (43) is used to eliminate one or another of the dimensionless products from (39), the solution becomes identical to (21), (22), or (23). Thus when the logarithmic law is introduced the solution is the same whichever procedure is followed. This occurs because the logarithmic law specifies that $\mathbf{V}$ and $\mathbf{V}_*$ have the same dimensions. If, however, we want a less restricted solution, valid regardless of the profile at the lower boundary, the generalized method gives the preferred result.

5. DISCUSSION

The foregoing analysis rests on two assumptions—that dimensional analysis is applicable to problems in which the relevant variables are two-dimensional vectors, or complex numbers, and that the equations of motion, together with the chosen set of boundary conditions, define a unique relationship between the wind and the stress. These assumptions will be discussed in turn.

From a mathematical point of view, generalized dimensional analysis may be thought of as the application of simple linear transformations which leave the governing equations unchanged. Since this procedure may be applied to any set of equations, scalar or otherwise, it appears that dimensional reasoning is indeed applicable to vector problems. From a physical point of view the question is less simple. We have seen that ordinary dimensional analysis cannot be applied to problems involving three-dimensional vectors. Yet many of the classical problems to which dimensional reasoning has been applied are essentially vectorial in nature, in that they involve forces and velocities which lie in different directions. Any problem involving a horizontal pressure gradient and a gravitational force is of this nature. Usually such problems contain certain basic symmetries which enable them to be handled by ordinary dimensional methods. However, Huntley [4] has suggested that in many cases more efficient solutions may be obtained when the dimensional procedure is modified by the introduction of three distinct length dimensions in the three orthogonal directions. For example, in the case of fluid flow through a pipe, the length dimension enters into a downstream velocity, the distance downstream from an orifice or a change in surface roughness, the pipe diameter, and the distance of a particle from a pipe wall. The first two obviously involve "downstream" distances while the last two involve "cross-stream" distances. According to Huntley's method, the ratio of distance from the wall to pipe diameter would be dimensionless while the ratio of distance from the orifice to pipe diameter would not. The same reasoning may be applied where three directions are involved. Huntley gives examples of problems which are intractable by the usual method but which may be solved when vector lengths are introduced.

It is not at all clear how this concept may be applied to atmospheric problems. In the problem considered in this paper, Huntley's method may be applied in a limited sense by considering two length dimensions, one in the horizontal (L_h) and one in the vertical (L_z). There is then no dimensional ambiguity; the horizontal velocities $\mathbf{V}$ and $\mathbf{V}_g$ have dimensions L_hT^{-1} while heights such as z, z_0 and z_h have the dimension L_z. The stresses $\boldsymbol{\tau}$, $\boldsymbol{\tau}_0$ then have the dimensions $ML_h^{-1}T^{-2}$, representing a force in a horizontal direction divided by a horizontal area, and $\mathbf{V}_*$, $\mathbf{V}_{*0}$ have the dimensions $L_h^{1/2}L_z^{1/2}T^{-1}$. (Note that in this context $\mathbf{V}/\mathbf{V}_*$ is *not* dimensionless.) Ordinary dimensional analysis then gives the same solution as was found by the generalized method, as is to be expected since this approach increases the number of dimensions by one. However, further progress through the introduction of a third length dimension appears impossible, for there is no way to ascribe dimensions to the ratio $\mathbf{V}/\mathbf{V}_g$ unless the two vectors are parallel or perpendicular. Suppose the wind is broken up into easterly and northerly components, u and v, having the dimensions L_xT^{-1} and L_yT^{-1} respectively, and the geostrophic wind is treated similarly. Then u/u_g and v/v_g are dimensionless, while u/v_g has dimensions $L_xL_y^{-1}$ and v/u_g has dimensions $L_yL_x^{-1}$. The ratio $\mathbf{V}/\mathbf{V}_g$ is dimensionless if both vectors are directed to the east, for example, and has dimensions $L_yL_x^{-1}$ if $\mathbf{V}$ is directed to the north and $\mathbf{V}_g$ to the east. But it is not clear what dimensions may be ascribed to $\mathbf{V}/\mathbf{V}_g$ if $\mathbf{V}$ is directed to the northeast and $\mathbf{V}_g$ to the east-southeast. It seems, therefore, that only problems containing certain fundamental symmetries (such as having all vectors lie in orthogonal directions) may be treated by the method of vector lengths, and that such problems may be adequately handled whether the vectors are two- or three-dimensional.

The question then arises of whether we are justified in ignoring the vector aspects of length and applying ordinary dimensional analysis to vector problems which do not contain such symmetries. We have seen that this method will not work when the vectors are three-dimensional (unless we introduce tensors, which are not generally considered in dimensional analysis), and there is therefore some doubt as to whether it is truly valid when the vectors are two-dimensional. At best it appears that a solution achieved in this way is less precise, in that it involves more dimensionless products, than one achieved by considering the vector nature of the problem, but as long as the complete three-dimensional vector problem cannot be handled the situation must be regarded as somewhat unsatisfactory. In presenting dimensional analysis as a special case of invariance theory, Long intimates that if more complex transformations than those mentioned earlier are applied, more detailed solutions can be found, and it may be that further development in this area will provide a basis for extending dimensional reasoning to general vector problems.

The question of whether equations (7) and (10) are sufficient to determine a unique solution of the form (9) is likewise difficult to answer. One may well argue that additional differential equations or boundary conditions may be needed. To a certain extent, the validity of the approach used is borne out by the general agreement of the result with the Rossby-Montgomery result; however this merely indicates that no serious discrepancy has arisen and cannot be taken as proof that the method is valid. Just as the validity of a solution found by ordinary dimensional analysis depends on the completeness of the set of relevant variables, so a solution found by the generalized method depends for its validity on the completeness of the set of governing equations, and it appears that in both instances we must rely largely on intuition to make the proper selection.

Several other points are worth noting. Although the ordinary and generalized procedures gave rise to different solutions when only equations (9) and (10) were used, both gave the same solution when use was made of the logarithmic law as a lower boundary condition. This simply means that in any atmosphere, if the form of the wind profile at the lower boundary is not known, the generalized approach gives the more efficient result. Thus if there is any doubt that the logarithmic law is the appropriate lower limit, the generalized solution (39) is to be preferred to the ordinary solution (14). When the wind profile near the ground is known, however, and is such that $\mathbf{V}$ and $\mathbf{V}_*$ must have the same dimensions, both procedures give the same result.

A second point is that although in principle the results achieved here may be subjected to experimental verification, in practice this may not be possible because of the difficulty of determining z_h, the total depth of the planetary layer. In fact it is questionable whether the clearly defined planetary boundary layer postulated here ever actually exists, except possibly under very special circumstances, and even then, it is not obvious how z_h might be measured.

Third, it must be emphasized that the development presented here refers to neutral conditions only. Under non-neutral conditions it is necessary to introduce additional governing equations involving the vertical heat flux and the associated buoyancy forces.

Fourth, one may argue that not all the variables appearing in (11) are independent, in that the surface stress is completely determined by the geostrophic wind, the surface roughness, the latitude, and the depth of the planetary layer. Indeed, this is the result stated in equation (19). However it seemed reasonable to deduce this result from the equations of motion, the boundary conditions, and the logarithmic law, rather than to assume it initially on an intuitive basis.

Finally, it must be noted that the "solutions" represented by equations (14), (19), and (39) are not complete in that they contain unspecified functions which must be determined by experiment. This is generally the case with solutions found by dimensional reasoning. The value of these solutions is that by specifying particular combinations of the relevant variables as being functionally related, they reduce the number of independent variables that must be considered in analyzing an experiment. The goal of this procedure is to achieve the greatest possible reduction in the number of variables, and therefore the dimensional method yielding the smallest number of dimensionless combinations is always to be preferred.

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REFERENCES

1. G. Birkhoff, *Hydrodynamics*, Princeton University Press, Princeton, 1950, 186 pp.
2. P. W. Bridgman, *Dimensional Analysis*, Yale University Press, New Haven, 1931, 113 pp.
3. E. Buckingham, "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations," *Physical Review*, vol. 4, 1914, pp. 354–376.
4. H. E. Huntley, *Dimensional Analysis*, MacDonald and Co., London, 1952.
5. D. C. Ipsen, *Units, Dimensions, and Dimensionless Numbers*, McGraw-Hill Book Co., Inc., New York, 1960.
6. H. L. Langhaar, *Dimensional Analysis and Theory of Models*, Wiley, N.Y., 1951, 166 pp.
7. R. R. Long, "The Use of the Governing Equations in Dimensional Analysis," *Journal of the Atmospheric Sciences*, vol. 20, No. 3, May 1963, pp. 209–212.
8. R. R. Long, *Engineering Science Mechanics*, Prentice-Hall, Englewood Cliffs, 1963, Chapter 9, pp. 365–387.
9. O. Reynolds, "On the Dynamical Theory of Incompressible Viscous Fluids and Determination of the Criterion," *Philosophical Transactions of the Royal Society of London*, Series A, vol. 186, 1895, pp. 123–164.
10. C.-G. Rossby and R. B. Montgomery, "The Layer of Frictional Influence in Wind and Ocean Currents," *Papers in Physical Oceanography and Meteorology*, vol. 3, No. 3, Massachusetts Institute of Technology and Woods Hole Oceanographic Institution, 1935, 101 pp.

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